

FREE AUTOMORPHISMS OF POSITIVE ENTROPY ON SMOOTH KÄHLER SURFACES

KEIJI OGUISO

ABSTRACT. We prove that there is a projective K3 surface admitting a (fixed point) free automorphism of positive entropy and that no smooth compact Kähler surface other than projective K3 surfaces and their blow up admits such an automorphism.

1. INTRODUCTION

Throughout this note, we work over the complex number field \mathbf{C} .

Let M be a smooth compact Kähler manifold and $g \in \text{Aut}(M)$ be a biholomorphic automorphism of M . The maximum $d_1(g)$ of absolute values of eigenvalues of the \mathbf{C} -linear extension of $g^*|H^2(M, \mathbf{Z})$ is called the *first dynamical degree* of g . By the fundamental result of Gromov-Yomdin ([Gr], [Yo], see also [Fr], Theorem (2.1) for explicit statement) and Dinh-Sibony ([DS], Corollary (2.2)), g is of *positive entropy* (resp. null entropy) if and only if $d_1(g) > 1$ (resp. $d_1(g) = 1$). We note that if g is not of null entropy, then g is of positive entropy by $\det g^*|H^2(M, \mathbf{Z}) = \pm 1$. We also note that $\dim M \geq 2$ and g is of infinite order if g is of positive entropy. Let

$$M^g := \{x \in M | g(x) = x\}.$$

g is said to be *free* if $M^g = \emptyset$. For instance, a non-trivial translation on a complex torus is free but it is of *null* entropy.

Both M^g and $d_1(g)$ are the most basic invariants in studying automorphisms of compact Kähler manifolds from complex dynamical point of view. For instance, McMullen ([Mc]) has found very impressive K3 surface automorphisms with Siegel disk. Both fixed point set giving candidates of the center of the Siegel disk and the first dynamical degree played crucial roles in his study.

Concerning these two invariants, Professor Shu Kawaguchi asked me the following:

Question 1.1. Are there smooth projective surfaces with free automorphisms of positive entropy?

The aim of this short note is to give the following answer to the question:

Theorem 1.2. (1) *Let S be a smooth compact Kähler surface with a free automorphism of positive entropy. Then S is birational to a projective K3 surface of Picard number ≥ 2 .*

(2) *Conversely, there is a projective K3 surface of Picard number 2 with a free automorphism of positive entropy.*

2010 *Mathematics Subject Classification.* 14J28, 14J50.

supported by JSPS Gran-in-Aid (B) No 22340009, JSPS Grant-in-Aid (S), No 22224001, and by KIAS Scholar Program.

We prove (1) in section 2. Our proof is based on a weaker version of the topological Lefschetz fixed point formula, a weaker version of the holomorphic Lefschetz fixed point formula (see eg. [GH], Chapter 3, section 4) and the following basic result of Cantat ([Ca], Proposition 1):

Theorem 1.3. *Let S be a smooth compact Kähler surface admitting an automorphism of positive entropy. Then S is bimeromorphic to either \mathbf{P}^2 , 2-dimensional complex torus, an Enriques surface, or a K3 surface.*

It may be also worth comparing (1) with the fact that a K3 surface admitting an automorphism with Siegel disk (hence of positive entropy and with a fixed point) is of algebraic dimension 0 ([Mc], see also [Og], Example (2.5)).

We shall prove (2) by using the global Torelli theorem for K3 surfaces (see eg. [BHPV], Chapter VIII, Theorem (11.1)) and the *golden number*

$$\eta := \frac{\sqrt{5} + 1}{2} .$$

Our construction (Theorem (4.1)) will be done in Section 4, based on a lattice-theoretic preparations in Section 3. See also Remarks (4.3), (4.4) for relevant results.

Acknowledgement. I would like to express my thank to Professor Shu Kawaguchi for valuable discussion and to Professors Mitsuhiro Shishikura and Hiroki Sumi for their encouragement and interest in this work.

2. SMOOTH COMPACT KÄHLER SURFACE WITH A FREE AUTOMORPHISM OF POSITIVE ENTROPY

In this section, we shall prove Theorem (1.2) (1).

Let M be a compact Kähler manifold of dimension n and $g \in \text{Aut}(M)$ be a biholomorphic automorphism of M . We define:

$$T(M, g) := \sum_{k=0}^{2n} (-1)^k \text{tr } g^* | H^k(M, \mathbf{Z}) ;$$

$$H(M, g) := \sum_{k=0}^n (-1)^k \text{tr } g^* | H^k(M, \mathcal{O}_M) .$$

Here $\text{tr } f$ is the trace of f . $T(M, g)$ is the *topological Lefschetz number* and $H(M, g)$ is the *holomorphic Lefschetz number*. It is not hard to see that $T(M, g)$ (resp. $H(M, g)$) is the integration of the Poincaré dual σ_Δ of the diagonal $\Delta \subset M \times M$ (resp. the sum of $(0, *)$ -th Kunneth components of σ_Δ) over the graph Γ_g of g (see eg. [GH], Pages 420-421, 423-424). Therefore, without more precise formula, we have:

Proposition 2.1. *If $M^g = \emptyset$, then $T(M, g) = L(M, g) = 0$.*

Let S be a smooth compact Kähler surface and $g \in \text{Aut}(S)$ be a biholomorphic automorphism of S .

Lemma 2.2. *Assume that S is bimeromorphic (in fact, birational) to either \mathbf{P}^2 or an Enriques surface. Then $S^g \neq \emptyset$.*

Proof. Note that $H^k(S, \mathcal{O}_S) = 0$ for $k \geq 1$. Thus

$$H(S, g) = \text{tr}(g^*|H^0(M, \mathcal{O}_S)) = 1 \neq 0.$$

Hence $S^g \neq \emptyset$. □

Lemma 2.3. *Assume that S is bimeromorphic to either 2-dimensional complex torus or a K3 surface. Let \bar{S} be the minimal model of S and $\pi : S \rightarrow \bar{S}$ be the naturally induced morphism. Then g descends to an automorphism \bar{g} of \bar{S} . Moreover, \bar{g} is of positive entropy (resp. free) if and only if g is of positive entropy (resp. free).*

Proof. The first assertion follows from the uniqueness of the minimal model of S with non-negative Kodaira dimension (see eg. [BHPV], Page 99, Proposition (4.6)). It is clear that g is free if so is \bar{g} . Conversely, if \bar{g} has a fixed point, say P , then g acts on $\pi^{-1}(P)$, which is either a point, say Q , or a tree of \mathbf{P}^1 . In the first case $g(Q) = Q$. In the second case, let C be the proper transform of the exceptional curve at the first blow up at P . Then $g(C) = C$ and g has at least one fixed point (as $C \simeq \mathbf{P}^1$). Hence \bar{g} is free if and only if so is g . Consider the following equivariant decomposition:

$$H^2(S, \mathbf{Z}) = \pi^*H^2(S, \mathbf{Z}) \oplus E$$

where E is the sublattice generated by the classes of exceptional divisors. The lattice E is of negative definite and $g^*(E) = E$. So, the eigenvalues $g^*|E$ are of absolute value 1. Hence $d_1(g) > 1$ if and only if $d_1(\bar{g}) > 1$. □

Lemma 2.4. *Assume that S is bimeromorphic to a 2-dimensional complex torus. Assume further that S has a free automorphism g . Then g is of null entropy.*

Proof. By Lemma (2.3), we may (and will) assume that S itself is minimal, i.e., S is a 2-dimensional complex torus. Then under the global linear coordinates $\mathbf{x} := (x_1, x_2)$ of the universal cover \mathbf{C}^2 of S , the automorphism g is written as in the following form:

$$g^*(\mathbf{x}) = A\mathbf{x} + \mathbf{b},$$

where $A \in \text{GL}_2(\mathbf{C})$ and $\mathbf{b} \in \mathbf{C}^2$. Let α and β be the eigenvalues of A counted with multiplicities. Then α and β are the eigenvalues of the action of g on $H^0(S, \Omega_S^1)$. By the Hodge duality together with the fact that $H^0(S, \Omega_S^2) = \wedge^2 H^0(S, \Omega_S^1)$, we obtain

$$H(S, g) = 1 - (\bar{\alpha} + \bar{\beta}) + \bar{\alpha}\bar{\beta} = (1 - \bar{\alpha})(1 - \bar{\beta}).$$

Since $H(S, g) = 0$, it follows that either $\alpha = 1$ or $\beta = 1$. Moreover, by

$$H^1(S, \mathbf{Z}) \otimes \mathbf{C} = H^0(S, \Omega_S^1) \oplus \overline{H^0(S, \Omega_S^1)},$$

it follows that the eigenvalues of (the \mathbf{C} -linear extension of) $g^*|H^1(S, \mathbf{Z})$ are $\alpha, \beta, \bar{\alpha}$ and $\bar{\beta}$ counted with multiplicities. Thus

$$\alpha\beta\bar{\alpha}\bar{\beta} = \pm 1,$$

because $g^*|H^1(S, \mathbf{Z}) \in \text{GL}(H^1(S, \mathbf{Z})) \simeq \text{GL}_4(\mathbf{Z})$. Hence if either $\alpha = 1$ or $\beta = 1$, then both $|\alpha| = 1$ and $|\beta| = 1$ hold. Since $H^2(S, \mathbf{Z}) = \wedge^2 H^1(S, \mathbf{Z})$, it follows that the eigenvalues of $g^*|H^2(S, \mathbf{Z})$ are of absolute value 1. Hence $d_1(g) = 1$. □

Lemma 2.5. *Assume that S is bimeromorphic to a K3 surface \bar{S} . Let ω_S be a generator of $H^0(\Omega_S^2) \simeq \mathbf{C}$. Assume further that S has a free automorphism g . Then $g^*\omega_S = -\omega_S$ and S is projective. Moreover $\rho(\bar{S}) \geq 2$ where $\rho(\bar{S})$ is the Picard number of \bar{S} .*

Proof. We use the same notation as in Lemma (2.3). Note that $H^1(\mathcal{O}_S) = 0$ and that $H^0(\Omega_S^2) = \mathbf{C}\omega_S$ is the Serre dual of $H^2(\mathcal{O}_S)$. Let $g^*\omega_S = \alpha\omega_S$. Then

$$0 = H(S, g) = 1 + \alpha^{-1}$$

and therefore $\alpha = -1$. This also implies that $\bar{g}^*\omega_{\bar{S}} = -\omega_{\bar{S}}$. Therefore \bar{S} is projective by Theorem (2.6) below. Since the induced automorphism \bar{g} of \bar{S} is free by Lemma (2.3), it follows that $T(\bar{S}, \bar{g}) = 0$. Let $\text{NS}(\bar{S})$ be the Néron-Severi lattice of \bar{S} and $T(\bar{S})$ be the transcendental lattice of \bar{S} , i.e.,

$$T(\bar{S}) = \{x \in H^2(\bar{S}, \mathbf{Z}) \mid (x, \text{NS}(\bar{S})) = 0\}.$$

Since \bar{S} is projective, it follows that the lattice $\text{NS}(\bar{S})$ is non-degenerate and therefore $\text{NS}(\bar{S}) \oplus T(\bar{S})$ is a subgroup of finite index of $H^2(\bar{S}, \mathbf{Z})$. We also note that $\text{rank NS}(\bar{S}) = \rho(\bar{S}) \geq 1$ (as \bar{S} is projective) and $\text{rank } T(\bar{S}) = 22 - \rho(\bar{S})$. Thus

$$0 = T(\bar{S}, \bar{g}) = 2 + \text{tr } \bar{g}^*|_{\text{NS}(\bar{S})} + \text{tr } \bar{g}^*|_{T(\bar{S})}.$$

Since $\bar{g}^*\omega_{\bar{S}} = -\omega_{\bar{S}}$, it follows that $\bar{g}^*|_{T(\bar{S})} = -\text{id}_{T(\bar{S})}$ by [Ni1], Theorem 0.1 (see also [Og], Theorem 2.4 (2)). So, if $\rho(\bar{S}) = 1$, then

$$T(\bar{S}, \bar{g}) = 2 + 1 - (22 - 1) = -20 \neq 0.$$

Therefore $\rho(\bar{S}) \neq 1$, whence $\rho(\bar{S}) \geq 2$. □

Theorem 2.6. *Let X be a compact hyperkähler manifold, that is, a smooth simply-connected compact Kähler manifold with an everywhere nondegenerate holomorphic 2-form ω_X such that $H^0(\Omega_X^2) = \mathbf{C}\omega_X$. Assume that X admits a bimeromorphic automorphism g such that $g^*\omega_X = \zeta_n\omega_X$, where $\zeta_n \neq 1$ and ζ_n is a root of unity. Then X is projective.*

Proof. This theorem is first proved by Nikulin ([Ni1], Theorem (0.1)) when X is a K3 surface and g is of finite order. Then this theorem is generalized by Beauville ([Be], Proposition 6, see also Proposition 7) when X is a hyperkähler manifold and g is a biholomorphic automorphism of finite order and by [Og], Theorem (2.4) when X is a hyperkähler manifold and g is a bimeromorphic automorphism (not necessarily of finite order). □

The following remark was the starting point of our construction in sections 3 and 4, even though it is no longer needed in the construction.

Remark 2.7. Let S be a K3 surface of Picard number 2 with a free automorphism g . Let $\varphi(t)$ be the characteristic polynomial of $g^*|_{\text{NS}(S)}$. Then $\varphi(t)$ is either $t^2 - 18t + 1$ or $t^2 - 18t - 1$. In particular, g is of positive entropy. Moreover, if $\varphi(t) = t^2 - 18t + 1$, then the eigenvalues of $g^*|_{\text{NS}(S)}$ are $9 \pm 4\sqrt{5} = \eta^{\pm 6}$, where η is the golden number (see Introduction for the definition of η).

Proof. Since $\rho(S) = 2$ and $g^*|_{T(S)} = -\text{id}_{T(S)}$ as observed in the proof of Lemma (2.5), it follows that

$$0 = H(S, g) = 2 + \text{tr } g^*|_{\text{NS}(S)} + (-1)(22 - 2).$$

Hence $\text{tr } g^*|_{\text{NS}(S)} = 18$. Since $g^*|_{\text{NS}(S)} \in \text{GL}(\text{NS}(S)) \simeq \text{GL}_2(\mathbf{Z})$, it follows that $\det(g^*|_{\text{NS}(S)}) = \pm 1$. Thus $\varphi(t) = t^2 - 18t \pm 1$, and one of eigenvalues of $g^*|_{\text{NS}(S)}$ is greater than 1. The last assertion can be checked by a straightforward calculation. □

Now we are ready to complete the proof of Theorem (1.2) (1). Let S be a smooth compact Kähler surface and g be a free automorphism of positive entropy. Since g is of positive entropy, S is bimeromorphic to either \mathbf{P}^2 , an Enriques surface, 2-dimensional complex torus or a K3 surface by Theorem (1.3). Since g is free, the first two cases are ruled out by Lemma (2.2). Since g is free but of positive entropy, the third case is also ruled out by Lemma (2.4). Hence S is birational to a projective K3 surface of Picard number ≥ 2 by Lemma (2.5). This proves Theorem (1.2)(1).

3. HYPERBOLIC LATTICES ARISING FROM THE GOLDEN NUMBER

In this section, we shall construct even hyperbolic lattices from the golden number η (see Introduction for the definition of η). The minimal polynomial of η over \mathbf{Z} is $t^2 - t - 1$,

$$N := \mathbf{Z}[\eta]$$

is the ring of algebraic integers of $\mathbf{Q}(\sqrt{5})$ and $\langle \eta \rangle \times \langle -1 \rangle$ is the unit group $\mathbf{Z}[\eta]^\times$ of N .

Lemma 3.1. *Let $\{a_n\}_{n \geq 0}$ be the Fibonacci sequence:*

$$a_{n+2} = a_{n+1} + a_n, \quad a_0 = 0, \quad a_1 = 1.$$

Then $\eta^n = a_n \eta + a_{n-1}$ for each positive integer n . For instance $\eta^3 = 2\eta + 1$, $\eta^4 = 3\eta + 2$, $\eta^5 = 5\eta + 3$, $\eta^6 = 8\eta + 5$, $\eta^7 = 13\eta + 8$.

Proof. Argue by induction on n using the fact that $\eta^2 = \eta + 1$. □

The elements $e_1 := 1$ and $e_2 := \eta$ form the free basis of N as \mathbf{Z} -module:

$$N = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2 \simeq \mathbf{Z}^2.$$

Let $n \in \mathbf{Z}$. Since $\eta \in \mathbf{Z}[\eta]^\times$, the homomorphism

$$\eta^n : N \rightarrow N ; \quad p(\eta) \mapsto \eta^n p(\eta)$$

is an automorphism of the \mathbf{Z} -module N .

Lemma 3.2. *Let n be a positive integer.*

(1) *The eigenvalues of the automorphism η^{2n} on N are η^{2n} and $1/\eta^{2n}$.*

(2) *The characteristic polynomial of η^{2n} on N is $t^2 - (a_{2n} + 2a_{2n-1})t + 1$, where $\{a_n\}_{n \geq 0}$ is the Fibonacci sequence. For instance, the characteristic polynomials of η^2 , η^4 , η^6 are $t^2 - 3t + 1$, $t^2 - 7t + 1$, $t^2 - 18t + 1$ respectively.*

Proof. The eigenvalues of η on N are

$$\frac{1 \pm \sqrt{5}}{2} = \eta, \quad \frac{-1}{\eta}.$$

Hence the assertion (1) follows. By Lemma (3.1), we have

$$\eta^{2n} = a_{2n}\eta + a_{2n-1}.$$

Taking the Galois conjugate $\eta \mapsto -1/\eta$, we obtain

$$\frac{1}{\eta^{2n}} = a_{2n} \frac{-1}{\eta} + a_{2n-1}.$$

Thus

$$\eta^{2n} + \frac{1}{\eta^{2n}} = a_{2n}(\eta + \frac{-1}{\eta}) + 2a_{2n-1} = a_{2n} + 2a_{2n-1} .$$

This together with (1) implies (2). \square

Let

$$b : N \times N \rightarrow \mathbf{Z}$$

be a \mathbf{Z} -valued symmetric bilinear form on N . With respect to the basis $\langle e_1, e_2 \rangle$, the form b is represented by the symmetric matrix

$$S_b := \begin{pmatrix} p & q \\ q & r \end{pmatrix} ,$$

where $p = b(e_1, e_1)$, $q = b(e_1, e_2)$ and $r = b(e_2, e_2)$. In terms of the matrix S_b , the form b even if p, r are even integers and b is hyperbolic if the matrix S_b is of signature $(1, 1)$. We call a pair (N, b) an *even hyperbolic lattice* if b is a \mathbf{Z} -valued even hyperbolic symmetric bilinear form on N . We have then a natural embedding

$$N \subset N^* := \text{Hom}_{\mathbf{Z}}(N, \mathbf{Z}) \subset N \otimes \mathbf{Q}$$

by b . The quotient group N^*/N is called the *discriminant group* of N .

An automorphism $f : N \rightarrow N$ as \mathbf{Z} -module is called an *isometry* of the lattice (N, b) if $b(f(x), f(y)) = b(x, y)$ for all $x, y \in N$. Note that an isometry f naturally induces an automorphism (as abelian group) of the discriminat group N^*/N .

We are interested in an even hyperbolic lattice structure (N, b) on our N such that η^2 , whence η^{2n} for all integer n , become isometries.

Proposition 3.3. *Assume that (N, b) is an even hyperbolic lattice and that η^2 is an isometry of (N, b) . Then:*

(1) *The matrix S_b is of the following form:*

$$S_b := \begin{pmatrix} 2q & q \\ q & -2q \end{pmatrix} ,$$

where q is a non-zero integer (and vice versa).

(2) *Under (1), the discriminant group N^*/N satisfies*

$$N^*/N = \langle \frac{e_2}{q} \rangle \oplus \langle \frac{e_1 + 2e_2}{5q} \rangle \simeq \mathbf{Z}/q \oplus \mathbf{Z}/5q .$$

(3) *Under (1), b does not represent 0, i.e., there is no $x \in N$ such that $b(x, x) = 0$. Moreover b does not represent ± 2 , i.e., there is no $x \in N$ such that $b(x, x) = \pm 2$ if and only if $q \neq \pm 1$.*

(4) *Under (1), η^6 acts on the discriminant group N^*/N as $-id_{N^*/N}$ if and only if q is one of $\pm 1, \pm 2$.*

Proof. Let $2p = b(e_1, e_1)$, $q = b(e_1, e_2)$ and $2r = b(e_2, e_2)$. By Lemma (3.1),

$$\eta^2(e_1) = \eta^2 = \eta + 1 = e_1 + e_2 ; \quad \eta^2(e_2) = \eta^3 = 2\eta + 1 = e_1 + 2e_2 .$$

Hence η^2 is an isometry if and only if

$$b(e_1, e_1) = b(e_1 + e_2, e_1 + e_2) , \quad b(e_1, e_2) = b(e_1 + e_2, e_1 + 2e_2) , \quad b(e_2, e_2) = b(e_1 + 2e_2, e_1 + 2e_2) ,$$

that is,

$$2p = 2p + 2q + 2r \text{ , } q = 2p + 3q + 4r \text{ , } 2r = 2p + 4q + 8r \text{ .}$$

This is equivalent to $r = -q$ and $p = q$. Now it is straightforward to see that S_b is hyperbolic if and only if $q \neq 0$. This proves (1).

By using (1), we compute that

$$b(e_1, \frac{e_2}{q}) = 1 \text{ , } b(e_2, \frac{e_2}{q}) = -2 \text{ ; } b(e_1, \frac{e_1 - 2e_2}{5q}) = 0 \text{ , } b(e_2, \frac{e_1 - 2e_2}{5q}) = 1 \text{ .}$$

Hence $e_2/q, (e_1 - 2e_2)/5q \in N^*/N$. Looking at the shape, we see that $e_2/q, (e_1 - 2e_2)/5q$ are of order $|q|$ and $|5q|$ respectively and $\langle e_2/q \rangle \cap \langle (e_1 - 2e_2)/5q \rangle = 0$ in N^*/N . Combining this with the fact that

$$|N^*/N| = |\det S_b| = 5q^2 \text{ ,}$$

we obtain (2).

Write $x = ae_1 + be_2$ ($a, b \in \mathbf{Z}$). Then

$$b(x, x) = 2q(a^2 + ab - b^2) \text{ .}$$

The result (3) follows from this equality.

Let us prove (4). By Lemma (3.1), we have $\eta^6 = 8\eta + 5$. $\eta^7 = 13\eta + 8$. That is,

$$\eta^6(e_1) = 5e_1 + 8e_2 \text{ , } \eta^6(e_2) = 8e_1 + 13e_2 \text{ .}$$

Thus

$$\eta^6(\frac{e_2}{q}) = \frac{8e_1 + 13e_2}{q} \text{ , } \eta^6(\frac{e_1 - 2e_2}{5q}) = -\frac{11e_1 + 18e_2}{5q} \text{ .}$$

Hence $\eta^6|_{N^*/N} = -id_{N^*/N}$ if and only if both

$$\frac{8e_1 + 14e_2}{q} \text{ , } \frac{2e_1 + 4e_2}{q}$$

are in N . This is equivalent to $q|2$. This proves (4). \square

Taking $q = 2$ in Proposition (3.3), we obtain the following result (see also Lemma (3.2) for the first statement of (2)):

Corollary 3.4. *Let (N, b) be the lattice (on our N) given by the matrix:*

$$S_b := \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix} \text{ ,}$$

i.e., the case where $q = 2$ in Proposition (3.3). Then:

- (1) (N, b) is an even hyperbolic lattice which represents neither 0 nor ± 2 .
- (2) η^6 is an isometry of (N, b) such that the characteristic polynomial is $t^2 - 18t + 1$ and the induced action on the discriminant group N^*/N is $-id_{N^*/N}$.

4. K3 SURFACES WITH A FREE AUTOMORPHISM OF POSITIVE ENTROPY

In this section, we shall prove Theorem (1.2) (2) by showing the following more explicit:

Theorem 4.1. *There exists a projective K3 surface S of $\rho(S) = 2$ such that $\text{NS}(S) = \mathbf{Z}h_1 \oplus \mathbf{Z}h_2$ where*

$$((h_i, h_j)) = \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}.$$

Any such K3 surface S (all of which form then a dense subset of 18-dimensional family) admits a free automorphism g of positive entropy.

Proof. Note that the abstract lattice given by the symmetric matrix above is an even hyperbolic lattice of rank 2. Hence the first result follows from [Mo], Corollary (2.9), which is based on the surjectivity of the period map for K3 surfaces (see eg. [BHPV], Page 338, Theorem 14.1) and Nikulin's theory ([Ni2]) of integral bilinear form.

Let us construct an automorphism g of S with desired properties.

Note that there is a lattice isomorphism $\varphi : \text{NS}(S) \simeq (N, b)$, where (N, b) is the lattice in Corollary (3.4).

Let $f := \varphi^{-1} \circ \eta^6 \circ \varphi$. Then f is an isometry of $\text{NS}(S)$. The eigenvalues of η^6 on N are η^6 and η^{-6} by Lemma (3.2). Thus so are the eigenvalues of f . Since both eigenvalues are positive, it follows that f preserves the positive cone, i.e., the connected component of

$$\{x \in \text{NS}(S) \otimes \mathbf{R} \mid (x, x) > 0\}$$

that contains ample classes. Since $\text{NS}(S) \simeq (N, b)$ does not represent -2 , the ample cone of S coincides with the positive cone. Thus f preserves the ample cone.

Since η^6 acts on the discriminant group N^*/N as $-id_{N^*/N}$, the isometry f also acts on the discriminant group $\text{NS}(S)^*/\text{NS}(S)$ as $-id_{\text{NS}(S)^*/\text{NS}(S)}$.

Let $T(S)$ be the transcendental lattice of S . Then $-id_{T(S)}$ also acts on the discriminant group $T(S)^*/T(S)$ as $-id_{T(S)^*/T(S)}$.

Hence the isometry $(f, -id_{T(S)})$ of $\text{NS}(S) \oplus T(S)$ extends to an isometry, say \tilde{f} , of $H^2(S, \mathbf{Z})$, by [Ni2], Proposition (1.6.1). Here we note that $H^2(S, \mathbf{Z})$ is unimodular and $\text{NS}(S)$ and $T(S) = \text{NS}(S)^\perp$ (in $H^2(S, \mathbf{Z})$) are both primitive in $H^2(S, \mathbf{Z})$. By construction, \tilde{f} preserves the Hodge decomposition of $H^2(S, \mathbf{Z})$ and, as observed above, also preserves the ample cone. Hence, by the global Torelli theorem for K3 surfaces (see eg. [BHPV], Chapter VIII Theorem (11.1)), there is an automorphism g of S such that $g^*|_{H^2(S, \mathbf{Z})} = \tilde{f}$.

Since $f = \tilde{f}|_{\text{NS}(S)}$ has an eigenvalue $\eta^6 > 1$, it follows that g is of positive entropy.

Let us show that g is free. Since $g \neq id_S$, the fixed point set S^g consists of at most finitely many complete curves and at most finitely many points.

We first show that S^g contains no complete curve. In fact, if $g(C) = C$ for some complete curve C , then the class $[C]$ would be an eigenvector of $g^*|_{\text{NS}(S)}$ with eigenvalue 1. However, the eigenvalues of $g^*|_{\text{NS}(S)} = f$ are η^6 and η^{-6} , none of which is 1.

Hence S^g consists of at most finitely many points, say n points counted with multiplicities. Then, by the topological Lefschetz fixed point formula, we have

$$n = T(S, g) = 2 + \text{tr } g^*|_{\text{NS}(S)} + \text{tr } g^*|_{T(S)}.$$

By $g^*|_{\text{NS}(S)} = f$ and by $\text{tr } f = \text{tr } (\eta^6|_N) = 18$ (by Corollary (3.4) (2)), we obtain

$$\text{tr } g^*|_{\text{NS}(S)} = 18 .$$

On the other hand, by $g^*|_{T(S)} = -id_{T(S)}$ and $\text{rank } T(S) = 20$ (by $\rho(S) = 2$), we obtain

$$\text{tr } g^*|_{T(S)} = -20 .$$

Hence

$$n = T(S, g) = 2 + 18 - 20 = 0 .$$

Hence $S^g = \emptyset$. This completes the proof. \square

Remark 4.2. Let (S, g) be as in Theorem (4.1). By the shape of $\text{NS}(S)$ and the fact that $\text{NS}(S)$ represents neither 0 nor ± 2 , it follows from [SD], Theorem (6.1) that S is realized as a smooth quartic surface in \mathbf{P}^3 . It seems extremely hard but highly interesting to write down explicitly the equation of S and the action of g in terms of the global homogeneous coordinates of \mathbf{P}^3 , for at least one of such pairs.

Remark 4.3. Let (S, g) be as in Theorem (4.1). Then

$$T(S, g^2) = 2 + \eta^{12} + \frac{1}{\eta^{12}} + 20 > 0 .$$

Therefore g^2 has a fixed point, say P . Put $Q = g(P)$. Then $Q \neq P$ (by $S^g = \emptyset$) and $g(\{P, Q\}) = \{P, Q\}$. Let \tilde{S} be a smooth surface obtained by the blow up of S at P and Q . Then g lifts to a free automorphism of \tilde{S} with positive entropy. So, there is also a “non-minimal” K3 surface with a free automorphism of positive entropy.

Remark 4.4. For each integer $n \geq 2$, there is an n -dimensional smooth compact Kähler manifold with a free automorphism of positive entropy. One of the “cheapest” way to construct may be as follows. Let (S, g) be as in Theorem (4.1) and \mathbf{P}^{n-2} be the projective space of dimension $n - 2$. Then the pair of product type

$$(S \times \mathbf{P}^{n-2}, g \times id_{\mathbf{P}^{n-2}})$$

satisfies the desired property. Needless to say, it would be more interesting to see if it is possible to construct free automorphisms of positive entropy “which do not come from lower dimensional pieces” for higher dimensional manifolds.

REFERENCES

- [BHPV] Barth, W., Hulek, K., Peters, C., Van de Ven, : *Compact complex surfaces*. Second enlarged edition. Springer Verlag, Berlin-Heidelberg (2004).
- [Be] Beauville, A., : *Some remarks on Kähler manifolds with $c_1 = 0$* , Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math. **39** Birkhäuser Boston, Boston, MA (1983) 1-26.
- [Ca] Cantat, S., : *Dynamique des automorphismes des surfaces projectives complexes*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999) 901–906.
- [DS] Dinh, T.-C., Sibony, N., : *Groupes commutatifs d’automorphismes d’une variété kählérienne compacte*, Duke Math. J. **123** (2004) 311–328.
- [Fr] Friedland, S., : *Entropy of algebraic maps*, Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993). J. Fourier Anal. Appl. (1995) Special Issue, 215–228.
- [GH] Griffiths, P., Harris, J., : *Principles of algebraic geometry*, Pure and Applied Mathematics. Wiley-Interscience, New York, 1978.
- [Gr] Gromov, M., : *On the entropy of holomorphic maps*, Enseign. Math. **49** (2003) 217–235.

- [Mc] McMullen, C. T., : *Dynamics on K3 surfaces: Salem numbers and Siegel disks*, J. Reine Angew. Math. **545** (2002) 201–233.
- [Mo] Morrison, D., : *On K3 surfaces with large Picard number*, Invent. Math. **75** (1984) 105–121.
- [Ni1] Nikulin, V. V., : *Finite groups of automorphisms of Kählerian K3 surfaces*, Trudy Moskov. Mat. Obshch. **38** (1979) 75–137.
- [Ni2] Nikulin, V. V., : *Integer symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979) 111–177.
- [Og] Oguiso, K., : *Bimeromorphic automorphism groups of non-projective hyperkähler manifolds – a note inspired by C. T. McMullen*, J. Differential Geom. **78** (2008) 163–191.
- [SD] Saint-Donat, B., : *Projective models of K3 surfaces*, Amer. J. Math. **96** (1974) 602–639.
- [Yo] Yomdin, Y., : *Volume growth and entropy*, Israel J. Math. **57** (1987) 285–300.

KEIJI OGUIO, DEPARTMENT OF MATHEMATICS, OSAKA UNIVERSITY, TOYONAKA 560-0043 OSAKA,
JAPAN AND KOREA INSTITUTE FOR ADVANCED STUDY, HOEGIRO 87, SEOUL, 130-722, KOREA
E-mail address: `oguiso@math.sci.osaka-u.ac.jp`